Equivalence of Three-dimensional Spacetimes

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A solution to the equivalence problem in three-dimensional gravity is given and a practically useful method to obtain a coordinate invariant description of local geometry is presented. The method is a nontrivial adaptation of Karlhede invariant classification of spacetimes of general relativity. The local geometry is completely determined by the curvature tensor and a finite number of its covariant derivatives in a frame where the components of the metric are constants. The results are presented in the framework of real two-component spinors in three-dimensional spacetimes, where the algebraic classifications of the Ricci and Cotton-York spinors are given and their isotropy groups and canonical forms are determined. As an application we discuss Gödel-type spacetimes in three-dimensional General Relativity. The conditions for local space and time homogeneity are derived and the equivalence of three-dimensional Gödel-type spacetimes is studied and the results are compared with previous works on four-dimensional Gödel-type spacetimes.
PACS: 04.20.Cv, 04.20.Jb

1. INTRODUCTION

The arbitrariness in the choice of coordinates is a basic assumption underlying general relativity theory. This hypothesis gives rise to the equivalence problem, namely the problem of deciding whether two spacetime metrics are different or are transformable one to another by a coordinate transformation. In other words, given two solutions of the field's equations, how to know whether they describe the same gravitational field? Furthermore, it can be difficult from a given metric to distinguish between real physical effects and those which depend only on the choice of coordinates. That is, the related question of how to decide whether certain effects have a physical origin or are due to the coordinate system used? A solution of the equivalence problem provides a complete and invariant characterization of the spacetime local geometry from which the answers to these questions can be obtained.

From the mathematical point of view, the solution to the equivalence problem of n-dimensional Riemannian manifolds goes back to Christoffel [1] and the best approach was developed by Cartan [2], which requires a comparison of curvature tensor components and their first $n(n+1)/2$ covariant derivatives. The development of computer algebra opened the way to the formulation of a procedure for testing equivalence of four-dimensional spacetimes in practice, that is, the Karlhede classification [3, 4]. Finally, algorithms using both the Newman-Penrose spinor formalism and the algebraic classification of the irreducible parts of the curvature spinor were developed, enabling the implementation of the practical procedure in a computer algebra suite called CLASSI [5, 6], based on the computer algebra system for General Relativity SHEEP [6, 7, 8, 9, 10]. For a review on the equivalence problem see [11, 12, 13] and the references therein. It should be mentioned that, in order to deal with torsion, the equivalence problem techniques were generalized to Riemann-Cartan spacetimes [14, 15] and implemented in a suite of computer algebra programs called TCLASSI [15, 16, 17], which is also based on SHEEP.

In this paper we present a solution to the equivalence problem of three-dimensional spacetimes and give a practical method to obtain a coordinate invariant description of local geometry, which is presented by using spinor formalism. The method requires a nontrivial adaptation of Karlhede's invariant classification of spacetimes of general relativity.

In the next section, we present a review Cartan's solution to the equivalence problem. In section 3, Kalhede's invariant classification of spacetimes is presented. In section 4,we present a brief review of the formalism of two-component real spinors in three-dimensional spacetimes including the curvature and the Cotton-York spinors, as well as the Ricci and Bianchi identities. In section 5, we show the algebraic classifications of the Ricci and Cotton-York spinors, including their canonical forms and isotropy groups, by using the spinor formalism. In section 6, we obtain a minimal set of components of the n-th derivatives of the Riemannian curvature spinor such that all derivatives of a given order m can be expressed algebraically (using sums, products and contractions) in terms of these sets for $n \leq m$. This new result given here for three-dimensional spacetimes is analogous to the result obtained by MacCallum and Åman [18] for four-dimensional spacetimes. In section 7, three-dimensional Gödel-type spacetimes are examined by using the equivalence problem techniques. The conditions for local space and time homogeneity are derived and the equivalence of three-dimensional Gödel-type spacetimes is studied. An invariant classification is obtained and the results are compared with previous works on four-dimensional Gödel-type spacetimes. Finally, in section 8 we present some conclusions.

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2. CARTAN'S SOLUTION

In this section we review Cartan's solution to the equivalence problem, applied to pseudo-Riemannian manifolds. The result is presented for n-dimensional manifolds. This is a very general result, since it is obtained for an arbitrary manifold without the use of any gravitational field's equations.

The (local) gravitational field in general relativity is completely determined by the components $g_{\mu\nu}$ of the metric tensor in a given coordinate system, which are a solution of Einstein's equations. Therefore, within the context of general relativity, (local) equivalence of spacetimes means (local) isometry of pseudo-Riemannian spacetimes.

In a more formal way, we say that two n-dimensional spacetimes M e $\widetilde{M}$, are (locally) equivalent when there exists a diffeomorphism $f : U \mapsto \widetilde{U}$ between two coordinate systems (U, x) and $(\widetilde{U}, \tilde{x})$, where $U \subset M$ and $\widetilde{U} \subset \tilde{M}$ are open sets defined on M and $\widetilde{M}$, respectively, such that $\tilde{x} = f(x)$ and

$$\tilde{g}_{\mu\nu}(\tilde{x}) = \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(x), \tag{2.1}$$

where $g_{\alpha\beta}(x)$ and $\tilde{g}_{\mu\nu}(\tilde{x})$ are the components of the metrics on M and $\widetilde{M}$ with respect to the coordinate systems (U, x) and $(\widetilde{U}, \tilde{x})$, respectively.

Despite the intuitive meaning of the equivalence definition given by eq. (2.1), its reformulation in terms of differential 1-forms is mostly desirable, taking into account Cartan's method to determine the equivalence of sets of 1-forms. Let $\omega^a = \omega^a{}_\mu dx^\mu$ and $\tilde{\omega}^a = \tilde{\omega}^a{}_\mu d\tilde{x}^\mu$ be non-holonomic coframes uniquely defined in coordinates systems (U, x) and $(\widetilde{U}, \tilde{x})$ on n-dimensional manifolds M and $\widetilde{M}$, respectively. We say that these sets of linearly independent 1-forms are equivalent when there exists a coordinate transformation $x^\mu = x^\mu(\tilde{x})$ such that $\tilde{\omega}^a = \omega^a$. Cartan [2] showed that these non-holonomic coframes are equivalent if, and only if, the system of algebraic equations obtained by the comparison of the non-holonomic objects $C^a{}_{pq}$ and $\tilde{C}^a{}_{pq}$ and their covariant derivatives according to

$$\begin{aligned}
\widetilde{C}^a{}_{pq}(\tilde{x}) &= C^a{}_{pq}(x), \\
\widetilde{C}^a{}_{pq;m_1}(\tilde{x}) &= C^a{}_{pq;m_1}(x), \\
&\vdots \\
\widetilde{C}^a{}_{pq;m_1\ldots m_{(p+1)}}(\tilde{x}) &= C^a{}_{pq;m_1\ldots m_{(p+1)}}(x),
\end{aligned} \tag{2.2}$$

is compatible, that is, there exists a solution $x^\mu = x^\mu(\tilde{x})$. Here, and in what follows, the covariant derivative is denoted by a semi-colon.

The non-holonomic objects and their covariant derivatives at each member of eqs. (2.2) are obtained by calculating the exterior derivatives

$$d\omega^a = \frac{1}{2} C^a{}_{pq}\, \omega^p \wedge \omega^q, \tag{2.3}$$

$$dC^a{}_{pq} = C^a{}_{pq;m}\, \omega^m, \tag{2.4}$$

$$dC^a{}_{pq;m} = C^a{}_{pq;mn}\, \omega^n \tag{2.5}$$

and successively. The covariant derivative of order $p+1$ is the lowest order derivative which is functionally dependent on the elements of the set $\{C^a{}_{pq}, C^a{}_{pq;m_1}, \cdots, C^a{}_{pq;m_1\ldots m_p}\}$, given by all lower derivatives up to order p. That is, the $p+1$ derivative is expressible in terms of its predecessors. Since each derivative either gives (at least) one new functionally independent function or is the last we need to consider, we obtain the limit $(p+1) \leq n$, taking into account that there exist at most n functionally independent functions on an n-dimensional manifold.

The most appropriate context to deal with the equivalence problem according to Cartan's method is the bundle $F(M)$ of generalized orthogonal frames defined over a pseudo-Riemannian manifold M. The frame bundle $F(M)$ is a differentiable manifold whose points are given by a pair: a point p of M and a generalized orthogonal frame defined on p. That is, a manifold given by $F(M) = \bigcup_{p\in M} F_p$, where F_p is the set of all generalized orthogonal frames defined at $p \in M$, called the fiber over p. These frames are given by the linearly independent vector fields $h_a = h_a{}^\mu(x)\partial_\mu$ $(a = 1, \ldots, n)$, where the components of the metric $\eta_{ab} = g(h_a, h_b) = g_{\mu\nu} h_a{}^\mu h_b{}^\nu$ are constant and given by a symmetric matrix $\eta = (\eta_{ab})$, with the appropriate signature [6].

The generalized orthogonal frames cannot be used to define equivalence, since they are not uniquely defined. There exist linear transformations $h_a \mapsto \Lambda_a{}^b h_b$ which leave invariant the components of the metric $\eta_{ab} = \Lambda_a{}^c\, \eta_{cd}\, \Lambda_b{}^d$. These transformations are called generalized rotations and form the group $O(n)$ with $\frac{1}{2}n(n-1)$ parameters [6, 19, 20].

Underlying Cartans approach to solve the equivalence problem there is the fact that equivalent n-dimensional pseudo-Riemannian spacetimes M and $\tilde{M}$ have equivalent bundles of generalized orthogonal frames $F(M)$ and $F(\tilde{M})$, respectively. Locally, $F(M)$ is the product $U \times O(n)$ of the subset $U \in M$ and the generalized orthogonal group $O(n)$. The fiber F_p over a point p of U is isomorphic to the generalized orthogonal group $O(n)$, since it is the set of all generalized orthogonal frames defined at p, which are related by generalized orthogonal transformations. Thus, the coordinates of $F(M)$ are given by the n coordinates $x = (x^a)$ of the point p and the $n(n+1)/2$ coordinates (parameters) $\xi = (\xi^A)$ of the generalized orthogonal group $O(n)$.

The crucial point for Cartans approach is the requirement of a set of uniquely defined linearly independent 1-forms. Since the freedom in the choice of generalized orthogonal frames in M is lost in $F(M)$, there is a uniquely-defined basis of the cotangent space $T^*_P(F(M))$, given by both the canonical 1-form $\Theta^A = H^A_\mu(x,\xi)dx^\mu$ and the

connection 1-form $\Sigma^A_{\ B} = \Gamma^A_{\ B\mu}(x,\xi)dx^\mu + \Gamma^A_{\ B\xi}(x,\xi)d\xi^A$ of $F(M)$. Therefore, we can reformulate the definition of (local) equivalence given by eq. (2.1) in the following way [1, 4, 6, 21, 22, 23]. Let M and $\widetilde{M}$ be two n-dimensional pseudo-Riemannian manifolds and $F(M)$ e $\widetilde{F}(\widetilde{M})$ the fiber bundles of generalized orthogonal frames over M and $\widetilde{M}$, respectively. We say that M and $\widetilde{M}$ are (locally) equivalent when there exists a local diffeomorphism $J : F(M) \mapsto \widetilde{F}(\widetilde{M})$ such that

$$J^*\widetilde{\Theta}^A = \Theta^A \quad \text{and} \quad J^*\widetilde{\Sigma}^A_{\ B} = \Sigma^A_{\ B} \tag{2.6}$$

hold. Here J^* is the pull-back map defined from J.

A solution to the equivalence problem for pseudo-Riemannian manifolds can then be obtained by using Cartans result on the equivalence of sets of 1-forms together with Cartan's equations of structure for a pseudo-Riemannian manifold. The solution can be summarized as follows [1, 4, 6, 21, 22, 23]. Two n-dimensional pseudo-Riemann manifolds M and $\tilde{M}$ are locally equivalent if and only if there exists a local diffeomorphism J between their corresponding generalized orthogonal frame bundles $F(M)$ and $F(\tilde{M})$, such that the following system of algebraic equations relating the components of the curvature tensor and its covariant derivatives

$$\begin{aligned} R^A_{\ BCD} &= \widetilde{R}^A_{\ BCD} \,, \\ R^A_{\ BCD;M_1} &= \widetilde{R}^A_{\ BCD;M_1} \,, \\ &\vdots \\ R^A_{\ BCD;M_1\ldots M_{p+1}} &= \widetilde{R}^A_{\ BCD;M_1\ldots M_{p+1}} \,, \end{aligned} \tag{2.7}$$

are compatible as equations in generalized orthogonal frame bundle coordinates (x^a, ξ^A). The $(p+1)$-th derivative of curvature is the lowest derivatives which is functionally dependent on all the previous derivatives. The number of functionally independent functions is at most the dimension of $F(M)$. Since there is at least one new functionally independent function at each order of derivative, it follows that $p+1 \leq n(n+1)/2$.

In association with the necessary and sufficient conditions for (local) equivalence given by eqs. (2.7), Cartan's solution also shows that all (local) metric properties of an arbitrary n-dimensional pseudo-Riemannian manifold are described in a comprehensive and unique way by the set

$$I_p = \{R^A_{\ BCD}\,, R^A_{\ BCD;M_1}\,, \ldots, R^A_{\ BCD;M_1\ldots M_p}\}, \tag{2.8}$$

whose elements are called Cartan's invariants, since they are invariant under coordinate transformations on the base manifold. But they depend on the orientation of the frame and change under generalized orthogonal rotations. The theoretical upper bound for the number of covariant derivatives to be calculated is $n(n+1)/2$.

In principle, we can use the set I_p of Cartan's invariants to obtain all (local) properties of a spacetime that can be obtained from the components $g_{\mu\nu}$ of the metric in a given coordinate system. There are several results where the Cartan's invariants are used to investigate (local) properties of spacetimes. In the context of General Relativity we have, for instance, the determination of the spacetime isometry group [24, 25], the investigation of limits of families of spacetimes [26] and the local degrees of freedom on a spacetime [27].

Concluding this section we review the results where the dimensions of the isometry group and its isotropy subgroup are obtained from Cartan's invariants.

We say that a vector field with components v^α in a given coordinate system, defines a local isometry on a pseudo-Riemannian manifold M if, and only if, the following conditions are satisfied:

$$\pounds_v g_{\mu\nu} = v_{\mu;\nu} + v_{\nu;\mu} = 0\,. \tag{2.9}$$

The eq. (2.9) above are called Killing's equations and their solutions are Killing vector fields. They are the generators of the isometry group and the maximal number of linearly independent Killing vectors is the dimension of the isometry group. There is an isotropy subgroup when there exist Killing vector fields which generate one-parameter groups of transformations which leave invariant the points of the spacetime manifold. They are the generators of the isotropy subgroup.

According to Cartan's solution to the equivalence problem, we can say that for each (local) isometry of a manifold M there exits, in bijective correspondence, a diffeomorphism on the bundle of generalized orthogonal frames $F(M)$ which preserve the set I_p of Cartan's invariants. Therefore, when I_p has k_p functionally independent Cartan's invariants, then the system of algebraic equations eqs. (2.7) has k_p linearly independent equations and its solution is a diffeomorphism on the frame bundle $F(M)$ which depends on $\frac{1}{2}n(n+1) - k_p$ arbitrary constants. Dealing separately with the t_p functions of the spacetime coordinates (x^μ) and the $m_p = k_p - t_p$ functions of the parameters (ξ^A) of the generalized orthogonal group $O(n)$, it can be shown that [2, 3] there exists an isometry group of dimension r, with an isotropy subgroup of dimension s and acting on a orbit of dimension d, where

$$s = \frac{1}{2}n(n-1) - m_p\,, \tag{2.10}$$

$$r = \frac{1}{2}n(n+1) - k_p = s + n - t_p\,, \tag{2.11}$$

$$d = r - s = n - t_p\,. \tag{2.12}$$

3. KARLHEDE CLASSIFICATION

In this section we show how the difficulties to deal with Cartan's solution of the equivalence problem in practice are considerably reduced by a procedure to test equivalence, based on an algorithm developed by Karlhede, where all calculations are performed on the spacetime

base manifold and the maximal order of the derivatives is reduced [3].

Cartan's solution of the equivalence problem has an important aspect to be considered in practice. At each order $q = 0, 1, \ldots, (p+1)$ of derivative of the curvature tensor, there are several properties which can be determined. Their comparison constitute necessary conditions to test the equivalence and can be used at each order q to establish a practical procedure. The test finishes whenever one of these conditions is not satisfied. Only when all necessary conditions for all orders $q = 0, 1, \ldots, (p+1)$ of derivatives are satisfied it is necessary to verify the consistency of the system of algebraic equations given by eqs. (2.7), that is, the necessary and sufficient conditions for equivalence. The relevance of this approach is evident from the practical point of view when considering that there is no procure which makes formally decidable the problem of verifying whether or not a system of algebraic equations has a solution [10].

Following this approach, we will describe the steps needed in order to present Karlhede's algorithm to test the equivalence. Initially it is necessary to handle separately the spacetime manifold coordinates x^μ and the parameters ξ^A of the generalized orthogonal group. This is done by calculating the Cartan's invariants in a section of the bundle of generalized orthogonal frames, that is, with respect to a given generalized orthogonal frame. Therefore, all Cartan's invariants are calculated on the base spacetime manifold and no more depend on the parameters of the group of generalized rotations. Thus, now the set

$$I_p = \{R^a{}_{bcd}, \ldots, R^a{}_{bcd;m_1 \ldots m_{(p+1)}}\} \tag{3.13}$$

is given by the components $R^a{}_{bcd}$ of the curvature on M and their covariant derivatives, with respect to the generalized orthogonal frame which define the local section of the frame bundle. Nevertheless, the dependence of Cartan's invariants on the parameters ξ^A of the generalized orthogonal group still can be verified through their behavior under generalized rotations.

The next step deals with the reduction, at each order of differentiation $q = 0, 1, \ldots, (p+1)$, of the dimension of the fiber bundle effectively used to test the equivalence. This is achieved by choosing a generalized orthogonal frame which is aligned with invariant directions determined by Cartan's invariants. With this choice the freedom of generalized rotations is reduced and the frame is fixed as much as possible. These invariant directions are determined by the algebraic classifications of the Cartan's invariants, which define canonical forms for each one of them. These generalized orthogonal frames are called standard or canonical frames [3]. This step has two important aspects from a practical point of view. First, along with the reduction of the dimension of the effective frame bundle the maximal order of derivatives is also reduced. Second, the algebraic classifications define a set of necessary conditions given by the algebraic types, their canonical forms and groups of isotropy.

The reduction of the dimension of the effective frame bundle at each order $q = 0, 1, \ldots, (p+1)$ of derivatives, can be determined from the isotropy group H_q of the set I_q. This is the group of generalized rotations which leave invariant the canonical forms of the elements of I_q. Thus, when two spacetimes have the same isotropy group H_q, the number of functionally independent functions of the parameters ξ^A are the same as well [6]. Therefore, the freedom of generalized rotations of the canonical frame is reduced each time the dimension of H_q is less than the dimension of $H_{(q-1)}$. Since H_q is a subgroup of $H_{(q-1)}$, the parameters of the generalized rotations which do not belong to H_q can be used to fix furthermore the canonical frame.

The next step take into account the fact that local equivalence also requires that the Cartan's invariants have the same dependence on the coordinates x^μ of the spacetime manifold. Thus, at each order $q = 0, 1, \ldots, (p+1)$ of derivative it is necessary that equivalent spacetime manifolds have the same numbers t_q of functionally independent functions of the coordinates x^μ in the elements of I_q.

Finally, according to Cartan's solution of the problem of equivalence, the Cartan's invariants must be calculated until a order $q = (p+1)$ of derivatives where the elements of $I_{(p+1)}$ are functionally dependent on the elements of I_p. Therefore, the steps of the practical procedure finish at an order q of derivative where not only the groups $H_{(q+1)}$ and H_q are the same, but also the numbers $t_{(p+1)}$ and t_p. The last step is to check the occurrence of these equalities.

All the steps of the practical procedure discussed above can be joined in an algorithm which starts by setting $q = 0$ and has the following steps [3]:

1. Calculate the set I_q, i.e., the derivatives of the curvature up to the q-th order.

2. Fix the frame, as much as possible, by putting the elements of I_q into canonical forms.

3. Find the frame freedom given by the isotropy group H_q of transformations which leave invariant the canonical forms of I_q.

4. Find the number t_q of functionally independent functions of spacetime coordinates in the elements of I_q, brought into the canonical forms.

5. If the isotropy group H_q is the same as $H_{(q-1)}$ and the number of functionally independent functions t_q is equal to $t_{(q-1)}$, then let $q = p+1$ and stop. Otherwise, increment q by 1 and go to step 1.

This procedure provides a discrete characterization of n-dimensional pseudo-Riemannian spacetimes, called Karlhede classification, in terms of the following properties: the set of canonical forms in I_p, the isotropy groups $\{H_0, \ldots, H_p\}$ and the number of independent functions $\{t_0, \ldots, t_p\}$.

To check the equivalence of two pseudo-Riemannian spacetimes the above discrete properties of their Karlhede's classifications are compared and only when they match is it necessary to determine the compatibility of eqs.(2.7).

An important result obtained with the application of Karlhede algorithm is the reduction of the maximal order of derivatives, given by $n(n+1)/2$ according to Cartan's theorem.

In the context of General Relativity, we have four-dimensional pseudo-Riemannian spacetimes. The canonical frame is fixed through the principal directions of the Weyl tensor, obtained by the Petrov's classification. Since the isotropy groups os the Petrov types (except type 0) have dimension $s \leq 2$, we obtain the limit $(p+1) \leq 7$. For conformally flat spacetimes, the Weyl tensor vanish (Petrov type 0) and the principal directions of the Ricci tensor are used instead. As the isotropy groups of the Segre types (except Segre 0) of the Ricci tensor have dimensions $s \leq 3$, we obtain the limit $p+1 \leq 8$ [10].

For three-dimensional spacetimes, the Weyl tensor vanishes identically and the canonical frame is aligned with the principal directions of the Ricci tensor. Now the Segre types (except type 0) have isotropy groups of dimensions $s \leq 1$, and we find the limit $p+1 \leq 5$.

The Karlhede algorithm was implemented by using spinor formalism to deal with equivalence in General Relativity, since it enables symmetries which are complicated in tensorial form to be expressed in a simple way. Since the same simplifications occurs with respect to spinors in three-dimensional spacetimes, in the next section we briefly present the two-component real spinors in three-dimensional spacetimes and obtain some results required to implement the Karlhede algorithm.

4. TWO-COMPONENT REAL SPINORS

We shall consider three-dimensional spacetimes described by a metric with signature $(-++)$. In this section we present some results obtained by using the formalism of two-component *real* spinors [28, 29, 30], which is analogous to the Newman-Penrose [19, 31, 32] formalism of two-component complex spinors in four-dimensional spacetimes. It should be mention that only one type of spinor index is required for three-dimensional space. The spin transformation are the elements of the group $SU(1,1)$ for two-component complex spinors and the group $SL(2,R)$ for two-component real spinors [28, 30].

An one-index spinor will be denoted by ψ^A or ψ_A and has two real components, where capital latin indices take the values $0, 1$. These indices will be raised and lowered by the Levi-Civita symbol $\epsilon_{AB} = -\epsilon_{BA}$ ($\epsilon_{01} = 1$), according to

$$\psi_A = \psi^B \epsilon_{BA}\,, \quad \psi^A = \epsilon^{AB}\psi_B\,. \tag{4.14}$$

The inner product is given by $\psi^A\phi_A = \epsilon^{AB}\psi_B\phi_A$ and the spin frame $\{o^A, \iota^A\}$ is normalized by $o_A\iota^A = 1$, where $o^A = (1,0)$ and $\iota^A = (0,1)$.

The 2-order symmetric real spinors ϕ_{AB} corresponds to vectors in three-dimensional spacetime M. There is a correspondence between a null frame of real vectors $\{k^a, m^a, n^a\}$ (or a Lorentz frame $\{t^a, x^a, z^a\}$) in M and a spin frame $\{o^A, \iota^A\}$ given through the real and symmetric connecting quantities $\sigma^a{}_{AB}$ according to

$$\begin{aligned} k^a &= (t^a + z^a)/\sqrt{2} = \sigma^a{}_{AB} o^A\, o^B = \sigma^a_{\ 00}\,, \\ n^a &= (t^a - z^a)/\sqrt{2} = \sigma^a{}_{AB}\iota^A\iota^B = \sigma^a_{\ 11}\,, \\ m^a &= x^a = \sigma^a{}_{AB}(o^A\iota^B + o^B\iota^A)/\sqrt{2} = \sqrt{2}\,\sigma^a_{01}\,, \end{aligned} \tag{4.15}$$

where $m^a = x^a$ and z^a are space-like, t^a is time-like, k^a and n^a are light-like. The correspondence between the spaces also requires the correspondence between their inner products

$$\begin{aligned} g_{ab} &= \sigma_a{}^{AB}\sigma_b{}^{CD} g_{ABCD} \\ &= -k_a\, n_b - k_b\, n_a + m_a\, m_b \\ &= -t_a\, t_b + z_a\, z_b + x_a\, x_b\,, \end{aligned} \tag{4.16}$$

$$\begin{aligned} g_{ABCD} &= \sigma^a{}_{AB}\, \sigma^b{}_{CD}\, g_{ab} \\ &= -\frac{1}{2}(\epsilon_{AC}\,\epsilon_{BD} + \epsilon_{AD}\,\epsilon_{BC}). \end{aligned} \tag{4.17}$$

Now we can show that, corresponding to the normalization $o_A\iota^A = 1$ of the spin frame, we have

$$k_a n^a = t_a t^a = -1, \quad m_a m^a = x_a x^a = z_a z^a = 1, \tag{4.18}$$

with all other contractions vanishing identically. In general, an arbitrary vector v^a in the null frame eqs. (4.15) above corresponds to a symmetric real spinor ϕ_{AB} according to

$$v^a = \sigma^a{}_{AB}\phi^{AB}\,, \quad \phi^{AB} = -\sigma_a{}^{AB} v^a. \tag{4.19}$$

Now, taking into account eqs. (4.16)-(4.17), we obtain

$$v_a v^a = g_{ab} v^b v^a = g_{ABCD}\phi^{CD}\phi^{AB} = -\phi_{AB}\phi^{AB}. \tag{4.20}$$

The spinor ϵ_{AB} fulfills the identity

$$\epsilon_{A[B}\,\epsilon_{CD]} = 0, \tag{4.21}$$

where square parentheses denote skew symmetrization. One consequence of the identity eq. (4.21) above is the identity

$$\phi_A\psi_B - \phi_B\psi_A = \epsilon_{AB}\phi_N\psi^N, \tag{4.22}$$

for any spinors ϕ_A and ψ_A, which will be used repeatedly to obtain the decomposition into irreducible parts of the curvature spinor.

For three-dimensional spacetimes the Weyl tensor vanishes identically and the Riemannian curvature tensor can be decomposed in terms of the Ricci tensor $R_{ab} = R^c{}_{acb}$ and the scalar of curvature R. Using the traceless

Ricci tensor $S_{ab} = R_{ab} - \frac{1}{3} g_{ab} R$, the Riemann tensor is given by [33]

$$R_{abcd} = g_{ac}S_{bd} - g_{ad}S_{bc} + g_{bd}S_{ac} - g_{bc}S_{ad} - \frac{1}{6}(g_{ad}g_{bc} - g_{ac}g_{bd})R. \quad (4.23)$$

and we can calculate the curvature spinor which, by using the identity eq. (4.22), is decomposed into irreducible parts in accordance with

$$R_{AXBYCZDW} = \epsilon_{XY} R_{ABCZDW} + \epsilon_{AB} R_{XYCZDW}, \quad (4.24)$$
$$R_{ABCZDW} = \epsilon_{ZW} Q_{ABCD} + \epsilon_{CD} Q_{ABZW}, \quad (4.25)$$
$$Q_{ABCD} = \Phi_{ABCD} - \frac{1}{3} g_{ABCD}\Lambda. \quad (4.26)$$

Therefore, the irreducible parts of the three-dimensional curvature spinor are the totally symmetric spinor $S_{ABCD} = 2\Phi_{ABCD}$, which corresponds to the traceless Ricci tensor S_{ab}, and the curvature scalar $\Lambda = R$. For the Ricci tensor R_{ab} we obtain the spinor

$$R_{ABCD} = 2\,\Phi_{ABCD} + \frac{1}{3} g_{ABCD}\Lambda. \quad (4.27)$$

Similar results have been obtained in [29], but with a different signature of the metric $(+\,-\,-)$ and a different choice of Λ in order to resemble the results for four-dimensional spacetimes.

Now we can express the Ricci spinor Φ_{ABCD} using the following abbreviations for the null frame components of the traceless Ricci tensor S_{ab}. We define a real symmetric Φ_{AB} $(A, B = 0, 1, 2)$ by

$$\Phi_{00} := \Phi_{0000} = \frac{1}{2} S_{ab}k^a k^b = \frac{1}{2} R_{ab} k^a k^b, \quad (4.28)$$
$$\Phi_{22} := \Phi_{1111} = \frac{1}{2} S_{ab}n^a n^b = \frac{1}{2} R_{ab} n^a n^b, \quad (4.29)$$
$$\Phi_{10} := \Phi_{1000} = \frac{1}{\sqrt{2}}(\frac{1}{2} S_{ab} m^a k^b) = \frac{1}{\sqrt{2}}(\frac{1}{2} R_{ab} m^a k^b), \quad (4.30)$$
$$\Phi_{12} := \Phi_{1011} = \frac{1}{\sqrt{2}}(\frac{1}{2} S_{ab} m^a n^b) = \frac{1}{\sqrt{2}}(\frac{1}{2} R_{ab} m^a n^b), \quad (4.31)$$
$$\Phi_{11} := \Phi_{0011} = \frac{1}{2} S_{ab} k^a n^b = \frac{1}{6}(R_{ab} m^a m^b + R_{ab} n^a k^b). \quad (4.32)$$

Note that Φ_{AB} has only five independent components, since we have the identity $\Phi_{11} \equiv \Phi_{02}$ due to $S = S^{a}_{\ a} = -2S_{ab}n^a k^b + S_{ab}m^a m^b = 0$ (this point is missing in [29]). When n^a and k^a are swapped, the index 1 in Φ_{AB} remains unchanged, while the rest flip $0 \leftrightarrow 2$. It should be mentioned that we follow Penrose-Rindler [31] and use the analogous identification between Φ_{AB} and Φ_{ABCD}, which is different from the choice used in [29].

Concluding, we present the Lorentz transformations performed in the null frame $\{k^a, m^a, n^a\}$ are given by [34, 35] boosts

$$\tilde{n}^a = \sqrt{A}\, n^a, \;\; \tilde{k}^a = \frac{1}{\sqrt{A}}\, k^a, \;\; \tilde{m}^a = m^a, \quad (4.33)$$

where $A > 0$, null rotations which leave n^a invariant

$$\tilde{k}^a = k^a + B\, m^a + \frac{1}{2} B^2 n^a, \;\; \tilde{m}^a = m^a + B\, n^a, \quad (4.34)$$

and null rotations which leave k^a invariant, given by eqs. (4.34) with n^a in place of k^a and conversely and a new parameter C replacing B, where $B, C \in R$. These transformations leave the components eqs. (4.16) of the metric invariant. The Lorentz group $SO(2,1)$ has three parameters.

In the next section, as required by the Kalhede classification, we review the algebraic classification of the Ricci and Cotton-York tensors in terms of Segre types, and obtain their canonical forms and the correspondent isotropy groups using the two-component real spinor formalism.

5. ALGEBRAIC CLASSIFICATION

The algebraic classification of the Ricci tensor in three-dimensional manifolds can be obtained by using a null triad and the freedom of Lorentz transformations in order to simplifying as much as possible their non-vanishing components [34, 35]. Thus, through Lorentz transformations it is possible to choose a null frame $\{k^a, m^a, n^a\}$, where $k^a n_a = -1$ and $m^a m_a = 1$, such that the Ricci tensor takes one of the following canonical forms:

Segre type	Canonical form	
$[11,1]$	$R_{ab} = -2\,\alpha\, k_{(a}n_{b)} - \beta(k_a k_b + n_a n_b) + \gamma\, m_a m_b$	(5.35)
$[(11),1]$	$R_{ab} = \gamma\, g_{ab} + (\gamma - \alpha)\,(k_a + n_a)(k_b + n_b)$	(5.36)
$[1(1,1)]$	$R_{ab} = \alpha\, g_{ab} + (\gamma - \alpha)\, m_a m_b$	(5.37)
$[(11,1)]$	$R_{ab} = \alpha\, g_{ab}$	(5.38)
$[1z\bar{z}]$	$R_{ab} = -2\alpha\, k_{(a}n_{b)} - \beta(k_a k_b - n_a n_b) + \gamma\, m_a m_b$	(5.39)
$[12]$	$R_{ab} = -2\alpha\, k_{(a}n_{b)} + \lambda k_a k_b + \gamma\, m_a m_b$	(5.40)
$[(12)]$	$R_{ab} = \alpha\, g_{ab} + \lambda k_a k_b$	(5.41)
$[3]$	$R_{ab} = \alpha\, g_{ab} + \mu(k_a m_b + m_a k_b)$	(5.42)

where $\alpha, \beta, \gamma \in R$ and $\beta \neq 0$ in eq. (5.39). It is possible to choose $\lambda = \pm 1$ and $\mu = \pm 1$. Note that the curvature scalar is given by $R = 2\,\alpha + \gamma$, except for Segre types [(11,1)], [(12)] and [3] where $R = 3\,\alpha$.

The above canonical forms correspond to the algebraic classification of R_{ab} in terms of Segre types $[1z\bar{z}]$, [12], [3] and [11,1], obtained through the solution of the eigenvalue problem

$$(R^a_b - \lambda g^a_b)v^b = 0, \tag{5.43}$$

where $\lambda \in C$.

Therefore, the canonical forms of the Ricci tensor determine null frames $\{m^a, n^a, k^a\}$, whose vectors are aligned with the principal directions of the Ricci tensor. These are the canonical frames of Karlhede procedure. They are not uniquely determined, since the Segre types and their degeneracies have isotropy groups whose elements are the Lorentz transformations which leave invariant the canonical forms.

By using the quantities defined by eqs.(4.28)–(4.32) and the canonical forms given by eqs.(5.35)–(5.42), we can obtain the following canonical forms and isotropy groups of the Segre types of the Ricci spinor, given in terms of the non vanishing Φ_{AB}:

Segre type	Canonical form	Isotropy group	
$[11,1]$	$\Phi_{00} = \Phi_{22},\, \Phi_{11}$	*none*	(5.44)
$[(11),1]$	$\Phi_{00} = \Phi_{22} = 3\,\Phi_{11}$	$SO(2)$	(5.45)
$[1(1,1)]$	Φ_{11}	$SO(1,1)$	(5.46)
$[(11,1)]$	$\Phi_{AB} = 0$	$SO(2,1)$	(5.47)
$[1z\bar{z}]$	$\Phi_{00} = -\Phi_{22},\, \Phi_{11}$	*none*	(5.48)
$[12]$	$\Phi_{22} = 1, \Phi_{11}$	*none*	(5.49)
$[(12)]$	$\Phi_{22} = 1$	$n.rot.(k_a inv)$	(5.50)
$[3]$	$\Phi_{12} = 1$	*none*	(5.51)

For Segre types [11,1] and [1z $\bar{z}$] the quantity Φ_{11} can be zero.

The Segre type $[1z\bar{z}]$, which is the only case to admit non-real eigenvalues, has three eigenvectors, two complex conjugate $k^a \pm in^a$ and one space-like m^a, with eigenvalues $\alpha \pm i\beta$ $(\beta \neq 0)$ and γ, respectively. The Segre type [12] has two eigenvectors, a null k^a and a space-like m^a, with eigenvalues α and γ, respectively. The Segre type [3] has one null eigenvector k^a with eigenvalue α [34, 35].

For Segre type [11, 1] the eigenvectors forms a Lorentz frame $t^a = (k^a + n^a)/\sqrt{2}$, $z^a = (k^a - n^a)/\sqrt{2}$ and $x^a = m^a$, whose eigenvalues are $\delta = \alpha + \beta$, $\rho = \alpha - \beta$ and γ, respectively. This is the only Segre type with time-like eigenvector. Therefore, it is possible to choose a Lorentz frame $\{t^a, x^a, z^a\}$ where the Segre type [11, 1] is given by the alternate form [34, 35]

$$R_{ab} = -\delta t_a t_b + \rho z_a z_b + \gamma x_a x_b. \tag{5.52}$$

The conformal properties of three-dimensional spacetime are described by the Cotton-York tensor [36]

$$C_{ab} = \sqrt{-g}\,\varepsilon_{bcd}\nabla^c (R^d{}_a - \frac{1}{4} g^d{}_a R), \tag{5.53}$$

where $g = det(g_{ab})$ and ε_{abc} is the Levi-Civita symbol with $\varepsilon_{012} = 1$. It is invariant under conformal transformations of the metric and vanishes for conformally flat spacetimes. It also satisfies the following conditions

$$C_{ab} = C_{ba}, \;\; C^a_{\ a} = 0, \;\; \nabla^b C_{ab} = 0 \tag{5.54}$$

of a symmetric, traceless, covariantly conserved tensor.

Since the Cotton-York tensor C_{ab} is symmetric and traceless, we find that the Cotton-York spinor Ψ_{ABCD} has the same symmetries as the Ricci spinor Φ_{ABCD}. Thus, it is classified through the same Segre types, with the same canonical forms given in terms of the quantities Ψ_{AB}, which are defined exactly as Φ_{AB}, i.e.,

$$\Psi_{00} := \Psi_{0000} = C_{ab}k^a k^b, \tag{5.55}$$
$$\Psi_{22} := \Psi_{1111} = C_{ab}n^a n^b, \tag{5.56}$$
$$\Psi_{10} := \Psi_{1000} = \frac{1}{\sqrt{2}} C_{ab}m^a k^b, \tag{5.57}$$
$$\Psi_{12} := \Psi_{1011} = \frac{1}{\sqrt{2}} C_{ab}m^a n^b, \tag{5.58}$$
$$\Psi_{11} := \Psi_{0011} = C_{ab}k^a n^b. \tag{5.59}$$
$$\tag{5.60}$$

Note that not only the Segre type, but also the principal spinors of Ψ_{ABCD}, may be different from those of Φ_{ABCD}.

At last we obtain a final result required in order to implement the Karlhede procedure using the most efficient way. It is not necessary to calculate all Cartan's invariants, since they are interrelated by both Bianchi and Ricci identities and their differential concomitants. In the next section, we use the spinor formalism to tackle the problem of specifying a minimal set of quantities to be computed at each step of differentiation of the Karlhede algorithm.

6. A COMPLETE MINIMAL SET OF N-TH CURVATURE DERIVATIVES

The first step of the Karlhede algorithm involve the computation of the covariant derivatives of the Riemann curvature. For economy in the computations, it is useful to specify a minimal set of quantities to be computed at each step of differentiation and we will discuss this problem here.

Thus, in this section Thus we wish to specify a minimal set of components of the spinor n-th derivatives of the Riemannian curvature such that all derivatives of a given order m can be expressed algebraically (using sums, products and contractions) in terms of these sets for $n \leq m$.

A relevant point to be taken into account when one needs to compute covariant derivatives of the curvature tensor is that they are interrelated by both Bianchi and Ricci identities and their differential concomitants.

The new result given here for three-dimensional spacetimes is the analogous to the result obtained by MacCallum and Åman [18] for four-dimensional spacetimes. Their result is given in terms of two-component complex

spinors and includes the Weyl spinor and its covariant derivatives. Here we use two-component real spinors but follows the same approach.

Although other minimal sets exist (and are related to our choice by the application of the Ricci and Bianchi identities) the chosen set has two nice properties: it is recursively defined (which avoids any need to compute additional n-th derivative quantities in order to find the higher derivatives) and contains only totally symmetric spinors (which simplifies storage and retrieval algorithms).

Before stating and proving the new result we review the derivation of the number of algebraically independent quantities that must be given to specify the invariants formed from n-th derivatives of the Riemannian curvature of a general three-dimensional spacetime (which is what the spinor components in a canonical frame are). This number is given by

$$\frac{3}{2}(n+4)(n+1). \tag{6.61}$$

The above result is most easily seen by considering invariants formed from the coordinate components of the $(n+2)$-th derivatives of the metric. Since partial derivatives commute, all such derivatives are expressible in terms of $g_{(ab),(c_1,c_2,\ldots,c_{(n+2)})}$, where round brackets denote symmetrisation. The number of these quantities can easily be computed by considering the partition of the indices in each symmetrisation between the three different possible values, giving

$$\frac{6(n+4)!}{2!(n+2)!} = 3(n+4)(n+3). \tag{6.62}$$

However, there are still the possible coordinate transformations to consider, specified by three functions whose $(n+3)$-th derivatives contribute to the $(n+2)$-th derivative of the metric. By a similar argument, there are

$$\frac{3(n+5)!}{2!(n+3)!} = \frac{3}{2}(n+5)(n+4) \tag{6.63}$$

distinct contributions arising in this way. The number of independent quantities stated above is the difference of eq. (6.63) and eq. (6.62).

The number of independent quantities is considerably less than the total number of components of the n-th derivative which is 6×3^n. One can easily compute the cumulative total number of these independent quantities up to the n-th derivative, which is

$$\frac{1}{2}(n+6)(n+2)(n+1). \tag{6.64}$$

For $n=6$ (the bound on p given by Cartan for $d=3$) the cumulative total number of all components of the derivatives is 4374, whereas the number just derived is 336. Similarly, for $n=5$ (the upper bound on p for metrics of Segre types [(11),1], [1(1,1)] and [(12)]) the numbers are 1458 and 231. Therefore, it is necessary a computer algebra implementation to carry out the calculations.

To end this part, we quote the spinor forms of the Bianchi identities

$$\nabla^{AB}\Phi_{ABCD} + \frac{1}{12}\nabla_{CD}\Lambda = 0 \tag{6.65}$$

and the Ricci identities

$$\nabla^N{}_{(A}\nabla_{B)N}\psi_C = \Phi_{ABCD}\psi^D + \frac{\Lambda}{6}(\epsilon_{CA}\psi_B + \epsilon_{CB}\psi_A). \tag{6.66}$$

Let us define the set of n-th derivatives $\nabla^n R$ to contain the following.

(i) The totally symmetrised n-th covariant derivatives of the Ricci spinor: $\nabla_{(AX}\nabla_{BW}\ldots\nabla_{GZ}\Phi_{HKLM)}$.

(ii) The totally symmetrised n-th covariant derivatives of the curvature scalar: $\nabla_{(AX}\nabla_{BW}\ldots\nabla_{GZ)}\Lambda$.

(iii) For $n\geq 1$ the totally symmetrised $(n-1)$-th covariant derivative of the Cotton-York spinor: $\nabla_{(AX}\nabla_{BW}\ldots\nabla_{GZ}\Psi_{HKLM)}$.

(iv) For $n\geq 2$, the dAlembertian of all quantities in $\nabla^{(n-2)}R$, i.e., $\Box\, Q \equiv \nabla^{NN}\nabla_{NN}\, Q$, where Q is a member of $\nabla^{(n-2)}R$.

The new result for three-dimensional spacetimes is obtained by following the same reasoning used by MacCallum and Åman [18], since we have similar relations between the quantities. Thus we have the following:

Theorem. All n-th derivatives of the Riemann tensor can be expressed algebraically in terms of the elements of $\nabla^r R$ for $0\leq r\leq n$ and this is a minimal such set of derivatives.

Proof. For $n=0$ the statement is merely the decomposition of the Riemannian curvature spinor and, for $n\geq 1$ the n-th derivatives of the curvature spinor are given by the n-th derivatives of the spinors Φ and Λ.

Following MacCallum and Åman, we use the notation $\sim$ for the equivalence relation that n-th derivatives are equal modulo algebraic expressions in the derivatives up to order $(n-1)$. The Ricci identity shows (see previous section) that for any spinor Q, the skew derivative

$$\begin{aligned}(\nabla^X{}_A\nabla^Y{}_B - \nabla^Y{}_B\nabla^X{}_A)Q &= \epsilon_{AB}\nabla^{(X}{}_C\nabla^{Y)C}Q \\ &\quad +\epsilon^{XY}\nabla_{Z(A}\nabla^Z{}_{B)}Q \\ &\sim 0 \end{aligned} \tag{6.67}$$

Consequently all n-th derivatives with the same indices on their differentiation operators, regardless of the order of these operators, are equivalent (under $\sim$). Moreover, a useful consequence of eq. (6.67) is

$$\nabla^X{}_C\nabla^{YC}Q \sim \epsilon^{XY}\nabla^{ZC}\nabla_{ZC}Q = \epsilon^{XY}\Box Q \tag{6.68}$$

The n-th derivatives can be decomposed into their totally symmetrised parts and products of the ϵ spinor with n-th derivatives having fewer but still symmetrised free indices. The totally symmetrised parts are covered by (i)-(ii) above, so we need consider only the parts involving contractions. We will use induction to prove that all such components are algebraically expressible using quantities of the forms (i)-(iv) above; the induction hypothesis is in effect used whenever we quote eq. (6.67) or eq. (6.68), since the equivalence uses terms from lower derivatives and we are assuming that these can be expressed in terms of quantities of the forms (i)-(iv) above.

The parts involving contractions are symmetric sums of terms, each of which involves a contraction. If the contraction indices belong to a pair of differentiation operators whose other indices are not contracted together, the term can be ignored as a result of eq. (6.67) or is converted by eq. (6.68) into a term involving contractions of both pairs of indices on a certain pair of differentiation operators (which will thus form a d'Alembertian). For this second type of contraction term, we can bring the d'Alembertian to the left (by eq. (6.67)); such terms will then be included in (iv) above. To complete the proof that (i)-(iv) are sufficient to represent all components of the n-th derivative of the Riemann curvature, we still have to prove that terms in which the only contractions are between differentiation operators and Riemann spinor indices are covered by (i)-(iii). We now consider this case.

Using eq. (6.67) we bring all those differentiation operators which are contracted with the Riemann spinor components to the right. We can ignore the Ricci scalar (since it has no indices on which to contract). By definition of Ψ, contraction of a differentiation operator with the Ricci spinor leads only to (derivatives of) Ψ and the contraction between the Ricci spinor and a differentiation operator has a symmetric part which is Ψ and a skew part which reduces, by the Bianchi identity eq. (6.65), to terms of the form (ii). The totally symmetrised derivatives of Ψ are (iii) above, so we have only to consider terms in which there is a contraction of a differentiation operator with Ψ_{ABCD}, that is, with $\nabla^{N}{}_{(A}\Phi_{BCD)N}$. We have now reduced the sufficiency proof to the consideration of (derivatives of) $\nabla^{A}{}_{F}\nabla^{N}{}_{A}\Phi_{BCDN}$ and $\nabla^{F}{}_{B}\nabla^{N}{}_{A}\Phi^{B}_{CDN}$. The first derivative, through the previous remarks about terms with contractions on differentiation operators, lead to quantities which are already included. The second derivative is equivalent, modulo d'Alembertian type terms, to a sum of terms of the form

$$\nabla^{F}{}_{B}\nabla^{N}{}_{A}\Phi^{B}{}_{CDN} = \nabla^{F}{}_{A}\nabla^{N}{}_{B}\Phi^{B}{}_{CDN} + \epsilon_{AB}\nabla^{FG}\nabla^{N}{}_{G}\Phi^{B}{}_{CDN} \quad (6.69)$$

the last expression following by the usual decomposition method. The first term on the right of eq. (6.69) reduces, by the Bianchi identity eq. (6.65), to a quantity of the form (ii) and the second reduces by eq. (6.67) and eq. (6.68) to a quantity of the form (iv).

Having proved that the set given by (i)-(iv) enables all n-th derivatives to be expressed, we have to show it is minimal. This is done simply by a counting argument. It is fairly easy to see, by a similar but simpler argument to that given above for counting invariants, that the constituent parts of $\nabla^n R$ as defined above contain respectively $2(n+1)+4$, $2(n+1)$, $[2(n-1)+1]+4$ and, assuming the induction hypothesis, $3(n+2)(n-1)/2$ real quantities. The total number of quantities in $\nabla^n R$ is therefore $3(n+4)(n+1)/2$ as required. This completes the proof of the theorem.

It should be mention that the theorem and the numbers of components to which we refer to above are for the general case.

In the next section, all these results are applied to study the properties of a three-dimensional spacetime. All the procedures were implemented using the *GrTensorII* package of *Maple* computer algebra system.

7. THREE-DIMENSIONAL HOMOGENEOUS GÖDEL-TYPE SPACETIMES

Gödels [37] solution of Einsteins field equations is the first cosmological model with rotating matter and closed time-like curves. It has shown that General Relativity does not forbid spacetimes with global causal pathologies. Nevertheless, it can represent rotating objects with physical meaning when surrounded by more standard spacetimes [38]. It should be mentioned that both time-like and null geodesics are not closed causal curves. The model is geodesically complete and has neither singularities nor horizons [20]. Due to its peculiarities, Gödel-type solutions have been studied with interest until nowadays with a fairly large literature.

Recently it has been shown that the three-dimensional Einstein-Maxwell theory with a cosmological constant and a Chern-Simons term have Gödel-type black holes and particle solutions [39]. Furthermore, it was shown that a one-parameter family of the three-dimensional Gödel-type metrics can be seen as arising from a deformation of anti-de Sitter metric, involving tilting and squashing of the lightcones. The anti-de Sitter metric appears as the boundary between the causal and non-causal models [40].

Actually, the four-dimensional Gödel spacetime metric has a direct product structure $ds^2_{(4)} = ds^2_{(3)} + dz^2$, where the three-dimensional metric $ds^2_{(3)}$ is a particular case of the Gödel-type line element, defined by

$$ds^2_{(3)} = -[dt + H(r)d\phi]^2 + D^2(r)d\phi^2 + dr^2. \quad (7.70)$$

All four-dimensional Gödel-type spacetimes that are homogeneous in space and time (hereafter ST homogeneous), are characterized by two parameter m^2 and ω, where

(i) $H = \omega\, r^2$, $D = r$, when $m = 0$;

(ii) $H = (2\,\omega/\mu^2)[1 - cos(\mu r)]$, $D = (1/\mu)sin(\mu r)$, when $m^2 = -\mu^2 < 0$;

(iii) $H = (4\,\omega/m^2) sinh^2(mr/2)$, $D = (1/m) sinh(mr)$, when $m^2 \geq 0$.

The constant ω is the vorticity of these rotating spacetimes. The Gödel spacetime is a particular case of the last class, for which $m^2 = 2\,\omega^2$, whose energy-momentum tensor $T_{\mu\nu}$ is given by

$$T_{\mu\nu} = \rho v_\mu v_\nu \, , \qquad v^\alpha = \delta^\alpha_{\ 0} \, , \tag{7.71}$$

$$\kappa\rho = -2\Lambda = m^2 = 2\,\omega^2 \, , \tag{7.72}$$

where κ and Λ are, respectively, the Einstein gravitational and the cosmological constants, ρ is the fluid density and v^α its four-velocity. The four-dimensional Gödel model has a 5-parameter group of isometries with an 1-parameter isotropy subgroup.

It should be mentioned that the three-dimensional Gödel-type line element eq. (7.70), besides being a solution of the Einstein-Maxwell-Chern-Symon theory, also satisfies the three-dimensional Einsteins equations for all values of (m^2, ω) and has 4-parameter group of isometries [39].

The (global) causality breakdown in ST homogeneous four-dimensional Gödel-type spacetimes depends on the behavior of $g_{\phi\phi} = D^2(r) - H^2(r)$, since the circles defined by $t, r, z = const$ are closed time-like curves when $g_{\phi\phi} < 0$ for a certain range of values of r. The causality features are the following [41]: (i) for $m^2 < 0$, there is an infinite sequence of alternating causal and non causal regions; (ii) for $0 \leq m^2 < 4\,\omega^2$, there is only one non causal region; (iii) for $m^2 \geq 4\,\omega^2$, there is no causality problem. Among these solutions there is the Rebouças-Tiomno solution [42], where $m^2 = 4\,\omega^2$, which is conformally flat, has a 7-parameter group of isometries and is not stably causal. All models where $m^2 > 4\,\omega^2$ are stably causal. Thus, the Reboças-Tiomno model is the boundary between the causal and non-causal models.

The problem of ST homogeneity of a four-dimensional spacetime endowed with a Gödel-type metric eq. (7.70) was investigated under restrictive assumptions on the form of the Killing vector fields [42, 43, 44]. The result was a set of necessary and sufficient conditions, which was rederived without assuming any simplifying hypothesis [45], by using a complete and invariant description of spacetimes obtained through the equivalence problem techniques.

Although the investigation of the causality features of three-dimensional Gödel-type spacetimes with parameters m^2 and ω can be investigated following the approach used for the four-dimensional case, the situation concerning the problem of ST homogeneity in its all generality is completely different. The necessary and sufficient conditions for ST homogeneity in 4-dimensions [45] were obtained by using a minimal set of independent covariant derivatives of the curvature spinor [18], calculated in a fixed canonical frame. Unfortunately this approach can not be used for three-dimensional manifolds, where the Weyl tensor vanishes identically [33], for the following reasons. First, it requires a different spinor formalism related to the Lorentz group $SO(2,1)$ of three-dimensional spacetimes. Second, since the minimal set [18] used depends on the Weyl spinor and its covariant derivatives, a new minimal set of independent covariant derivatives of the curvature spinor must be find. Finally, since the standard frame was fixed by aligning the frame vectors with the principal directions of the Weyl spinor, a different canonical frame fixed by the principal directions of the Ricci spinor is required.

We show that the ST homogeneous three-dimensional Gödel-type manifolds have a 4-parameter group of isometries and an 1-parameter subgroup of isotropy. The equivalence of these spacetimes is discussed and they are found to be characterized by two essential parameters m^2 and ω: identical pairs $(m^2,\, \omega)$ correspond to equivalent (isometric) manifolds. The algebraic classifications and canonical forms of both the Ricci spinor Φ_{AB} and the conformal Cotton-York spinor Ψ_{AB} are presented, by using the formalism of two-component real spinors [28, 29, 30]. For a general pair $(m^2,\, \omega)$, the Cotton-York spinor Ψ_{AB} and the Ricci spinor Φ_{AB} are both Segre type [(11),1]. The only exceptions are the conformally flat spacetimes given by the anti-de Sitter spacetime ($m^2 = 4\,\omega^2$), where $\Phi_{AB} = 0$ and $\Psi_{AB} = 0$, and the spacetime without rotation ($m^2 \neq 0$, $\omega = 0$), where $\Psi_{AB} = 0$. The group of isometries is also discussed. Finally, the results obtained are compared with those for the four-dimensional Gödel-type ST homogeneous metrics [45].

In this section we shall consider a three-dimensional pseudo-Riemannian manifold M, endowed with a Gödel-type metric eq. (7.70).

For arbitrary functions $H(r)$ and $D(r)$, the Gödel-type three-dimensional metrics are Segre type [11,1] and the canonical frame is completely fixed. The canonical frame is found in the following way. First, we calculate the Ricci spinor Φ_{AB} in the null triad

$$\begin{aligned} \theta^1 &= \omega^1, \\ \theta^2 &= (\omega^3 - \omega^2)/\sqrt{2}, \\ \theta^3 &= (\omega^3 + \omega^2)/\sqrt{2}, \end{aligned} \tag{7.73}$$

where ω^A is a Lorentz triad ($\eta_{AB} = diag(+1, +1, -1)$) given by

$$\omega^1 = dr, \;\; \omega^2 = D(r) d\phi, \;\; \omega^3 = dt + H(r) d\phi. \tag{7.74}$$

We find the non vanishing components Φ_{00}, Φ_{22} and Φ_{11}. In order to obtain the canonical form for the Segre type [11,1], where $\Phi_{00} = \Phi_{22}$, we perform a boost with parameter A(r) whose effect on the null frame above can be stated as

$$\tilde{\theta}^1 = \theta^1, \;\; \tilde{\theta}^2 = \sqrt{A(r)}\, \theta^2, \;\; \tilde{\theta}^3 = \frac{\theta^3}{\sqrt{A(r)}} \tag{7.75}$$

The canonical frame for the Segre type [11,1] is obtained

by choosing

$$A(r)^2 = \frac{\frac{D''}{D} - (\frac{H'}{D})^2 - (\frac{H'}{D})'}{\frac{D''}{D} - (\frac{H'}{D})^2 + (\frac{H'}{D})'} \qquad (7.76)$$

where the prime denotes derivative with respect to r.

Using the canonical frame above one finds the following non vanishing components of the Cartan's scalars, which correspond to the first step for q = 0 of our algorithm:

$$\begin{aligned} \Phi_{00} &= \Phi_{22} \\ &= \frac{A(r)}{4}\left[\frac{D''}{D} - \left(\frac{H'}{D}\right)^2 - \left(\frac{H'}{D}\right)'\right], \qquad (7.77) \\ \Phi_{11} &= -\frac{1}{12}\left[\frac{D''}{D} - \left(\frac{H'}{D}\right)^2\right], \qquad (7.78) \\ \Lambda &= -2\left[\frac{D''}{D} - \frac{1}{4}\left(\frac{H'}{D}\right)^2\right]. \qquad (7.79) \end{aligned}$$

For ST homogeneity one finds, from equation eq. (2.12), that we must have $t_p = 0$, that is, the number of functionally independent functions of the spacetime coordinates in the set I_p must be zero. Accordingly, all the above quantities of the minimal set must be constant. Thus, from eqs. (7.77)–(7.79) one easily concludes that for a three-dimensional Gödel-type spacetime metric eq. (7.70) to be ST homogeneous it is necessary that

$$\frac{H'}{D} = \text{const} \equiv 2\,\omega, \quad \frac{D''}{D} = \text{const} \equiv m^2. \qquad (7.80)$$

We shall now show that the above necessary conditions are also sufficient for ST homogeneity. Indeed, under the conditions eqs. (7.80) we obtain that $A(r) = 1$ and the non-vanishing Cartan's scalars corresponding to the first step for $q = 0$ of our algorithm reduce to

$$\begin{aligned} \Phi_{00} &= \Phi_{22} = 3\,\Phi_{11} = -\frac{1}{4}(m^2 - 4\,\omega^2) \qquad (7.81) \\ \Lambda &= -2\,(m^2 - \omega^2) \qquad (7.82) \end{aligned}$$

where obtain the canonical form $\Phi_{00} = \Phi_{22} = 3\,\Phi_{11}$ for Segre type [(11),1]. Therefore, we group the Gödel-type three-dimensional spacetimes, according to the relevant parameters m^2 and ω, into three classes:

(i) $m^2 \neq 4\,\omega^2$, where $m^2, \omega \neq 0$;

(ii) $m^2 = 4\,\omega^2$, where $\omega \neq 0$;

(iii) $m^2 \neq 0$, $\omega = 0$.

Now we proceed by carrying out the next steps of the procedure for testing equivalence in practice, for each class of Gödel-type three-dimensional spacetimes.

For the first class, we have that all metrics are Segre type [(11),1]. Following the algorithm of the previous section, one needs to find the isotropy group which leaves the above Cartan's scalars (canonical forms) invariant. The curvature scalar Λ is invariant under the whole Lorentz group $SO(2,1)$. Thus, the isotropy subgroup H_0 is determined by the Ricci spinor, which is Segre type [(11),1] and is invariant under the group $SO(2)$ of spatial rotations. So, we obtain that $t_0 = 0$ and that the residual group H_0, which leaves the above Cartan's scalars invariant, is one-dimensional.

We proceed by carrying out the next step of our practical procedure, i.e., by calculating the Cotton-York spinor and the totally symmetrised covariant derivatives of the Cartan's scalars eqs. (7.81)–(7.82), that is, the first step for $q = 1$. One finds the following non-vanishing quantities:

$$\Psi_{00} = \Psi_{22} = 3\,\Psi_{11} = -\frac{3}{4}\omega(m^2 - 4\,\omega^2), \qquad (7.83)$$

where the Cotton-York spinor is in the canonical form of Segre type [(11),1].

As no new functionally independent function arose, $t_0 = t_1$. Besides, the Cartan's scalars eq. (7.83) are invariant under the same isotropy group (spatial rotations), i.e. $H_0 = H_1$, and the algorithm stops. Thus we obtain $t_p = 0$ and 1-dimensional isotropy subgroup $H_p = SO(2)$. From eqs. (2.11)-(2.12) one finds that the three-dimensional Gödel-type spacetime have a four-dimensional group of isometries with a three-dimensional orbit – the necessary conditions eqs. (7.80) are also sufficient for ST homogeneity.

For the next class ($m^2 = 4\,\omega^2, \omega \neq 0$), following the algorithm, we obtain that

$$\Phi_{AB} = 0, \qquad (7.84)$$

$$\Lambda = -\frac{3}{2}m^2. \qquad (7.85)$$

Thus $t_0 = 0$ and $dim(H_0) = 3$, since the group of invariance of Φ_{AB} is now the Lorentz group $SO(2,1)$. Considering that the Cotton-York spinor $\Psi_{AB} = 0$ and that all derivatives of the Cartan's scalars vanishes identically, the process terminate. This is the anti-de Sitter three-dimensional spacetime. It is conformally flat and has a 6-dimensional isometry group with a three-dimensional isotropy subgroup and acts on a three-dimensional orbit. Therefore, according to eq. (2.12), it is ST homogeneous. Note that the four-dimensional spacetime with the same values of m^2, ω is the conformally flat Rebouças-Tiomno spacetime with a 7-parameter isometry group.

Finally, for the last class ($m^2 \neq 0, \omega = 0$), that is, the Gödel-type spacetimes without rotation, we obtain for the first step of the algorithm for q = 0 that

$$\begin{aligned} \Phi_{00} &= \Phi_{22} = 3\,\Phi_{11} = -\frac{1}{4}m^2 \qquad (7.86) \\ \Lambda &= -2m^2 \qquad (7.87) \end{aligned}$$

where Φ_{AB} is in canonical form for Segre type [(11),1]. Therefore, we find again that $t_0 = 0$ and that H_0 is the group $SO(2)$ of spatial rotations.

Following the algorithm, we obtain once more that the Cotton-York spinor $\Psi_{AB} = 0$ and that all derivatives vanishes identically, and the process terminate. Since $t_p = 0$ and $H_p = SO(2)$, we obtain again a 4-parameter isometry group with a 1-dimensional isotropy subgroup. Therefore, it is a ST homogeneous conformally flat spacetime. Notice that the four-dimensional Gödel-type spacetime without rotation is not conformally flat and has a 6-dimensional isometry group.

The above results can be summarized in the following theorems:

Theorem 1 *The necessary and sufficient conditions for a three-dimensional Gödel-type spacetime to be ST (locally) homogeneous are those given by equations eqs. (7.80).*

Theorem 2 *All ST locally homogeneous three-dimensional Gödel-type spacetimes admit a four-dimensional group of isometries with an 1-parameter isotropy subgroup and are characterized by two independent parameters m^2 and ω: identical pairs (m^2, ω) specify equivalent (isometric) spacetimes.*

It is worth emphasizing that eqs. (7.81)-(7.87) are related to the corresponding equations for four-dimensional Gödel-type spacetimes (eqs. (3.12)-(3.15) and eqs. (3.18)-(3.25) in [45]). Therefore, our study here can be considered as the lower dimensional analogous of results in [45]. Thus, for example, the above theorems 1 and 2 relates to the corresponding theorems in [45] (theorems 1 and 2 on page 891).

For all three-dimensional ST homogeneous Gödel-type spacetimes the Segre type of the Ricci spinor is [(11),1]. The only exception is the anti-de Sitter spacetime, obtained when $m^2 = 4\,\omega^2$, whose isometry group is 6-dimensional. This is the only condition where the isometry group has dimension higher than four. The Cotton-York spinor has also the same Segre type [(11),1] for all Gödel-type, except when the spacetime is anti-de Sitter ($m^2 = 4\,\omega^2$) and when the spacetime has no rotation ($\omega = 0, m^2 \neq 0$), since both are conformally flat ($\Psi_{AB} = 0$).

For the sake of completeness, we should mention that the equivalence problem techniques were used to investigate the five-dimensional Gödel-type and generalized Gödel-type pseudo-Riemannian spacetimes [46, 47]. Furthermore, Gödel-type solutions with torsion were also investigated [48] through the equivalence problem techniques for Riemann-Cartan spacetimes [14, 15, 17].

To conclude, we should like to emphasize that as no field equations were used to show the above results, they are valid for every three-dimensional pseudo-Riemannian Gödel-type solution regardless of the theory of gravitation considered.

8. FINAL REMARKS

It should be noted that the equivalence problem techniques, which we have used in this work, can certainly be used in more general contexts. Among possible applications we mention, especially, that the equivalence problem techniques for both general relativity and three-dimensional gravitation can be used to investigate their interconnections in a coordinate invariant way.

Acknowledgments

F. C. Sousa gratefully acknowledge financial assistance form CAPES. C. Romero would like to thank CNPq for partial financial support. Thanks also go to the referees for their suggestions and constructive comments.

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